INTRODUCTION

In this paper, we study the oscillatory and asymptotic behavior of solution of fourth order nonlinear delay difference equation of the form

\[ \Delta \left( \frac{1}{a_n} \left( \Delta^1 (y_n - P_n Y_{n-\sigma}) \right) \right) - q_{n+1} f(y_{\sigma(n)}) \leq 0 \]  

(1)

Here \( \Delta \) is the forward difference operator and defined by \( \Delta y_n = y_{n+1} - y_n \) where \( k \) is a fixed nonnegative integer and \( \{a_n\}, \{p_n\} \) and \( \{q_n\} \) are sequence of nonnegative integers with respect to the difference equation (1) throughout. A nontrivial solution \( \{y_n\} \) of equation (1) is said to be oscillatory if for any \( N \geq n \) there exists \( n > N \) such that \( y_n, y_{n+1} \leq 0 \). Otherwise, the solution is said to be non-oscillatory (Agarwal, 1992; Artzrouni, 1985; Cheng and Patula, 1993; Peterson, 1995; Philos and Purnaras, 2001) We shall assume that the following conditions hold:

\( (c_1): (\sigma_n, p_n) \) and \( (q_n) \) are real sequences and \( a_n \leq 0 \) for infinitely many values of \( n \).

\( (c_2): f: \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( y f(y) > 0 \) for all \( y \neq 0 \).

\( (c_3): \sigma (n) \geq 0 \) is an integer such that \( \lim_{n \to \infty} \sigma(n) = \infty \)

\( (c_4): R_n = \sum_{i=n}^{n} a_i \rightarrow \infty \) as \( n \rightarrow \infty \)

MAIN RESULTS

Theorem 1

In addition to the conditions

\( (c_1), (c_2), (c_3), (c_4) \), if the conditions are

\( (H1)\ P_n \geq 0 \) and \( \sum_{i=n}^\infty p_i = \infty \)

\( (H2)\ \lim_{(u \to \infty)} \inf |f(u)| \geq 0 \)

\( (H3)\ \lim_{u \to \infty} f(u) \geq 0 \)

Then every solution of equation (1) is oscillatory.

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Proof

Suppose that the equation (1) has non-oscillatory solution \( \{y_n\} \) is eventually positive. Then there is a positive integer \( n_0 \) such that \( y_{n_0} \geq 0, f \) or \( n \geq n_0 \) implies that \( \{y_n\} \) is non-oscillatory. Without loss of generality we can assume that there exists an integer \( n_1 \geq n_0 \) such that

\[ y_n > 0, \Delta y_n > 0, y_{n-m} \geq 0, \Delta y_{n-m} \geq 0 \quad \text{for all} \quad n \geq n_1. \]

Set then

\[ z_n = 0, \Delta z_n \geq 0, \quad \text{for all} \quad n = n_1 \]

From equation (1) we have

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) = q_{\sigma(n)} f(y_{\sigma(n)}) \quad \text{for all} \quad n \geq n_1 \]

In view of the conditions (c2), (c3), (H2) and from the equation (2), we obtain

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) \quad \text{is eventually non-increasing.} \]

We first show that

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) \leq -k_i \quad \text{for all} \quad n \geq n_2 \]

Summing the inequality (3) from \( n_2 \) to \( n - 1 \) we have

\[ \frac{1}{a_n} \Delta^2 z_n \leq -k_i (n-n_2) \quad \text{for all} \quad n \geq n_2 \]

Therefore

\[ \frac{1}{a_n} \Delta^2 z_n \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty \]

Then there exists an integer \( n_2 \geq n_1 \) and \( k_i > 0 \) such that

\[ \frac{1}{a_n} \Delta^2 z_n \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty \]

Summing the inequality (5) from \( n_1 \) to \( n - 1 \), we have

\[ \Delta^2 z_n \leq - k_i \sum_{n=1}^{n-1} a_n \quad \text{as} \quad n \rightarrow \infty \]

In view of the condition (c3), and from the inequality (6), we obtain

\[ \Delta z_n \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty \]

This shows that

\[ \frac{1}{a_n} \Delta^2 z_n \quad \text{is eventually non-increasing.} \]

For all large \( n \).

Let \( L = \lim_{n \rightarrow \infty} y_n \). Then \( L \) is finite or infinite.

Case 1

\( L > 0 \) is finite.

In view of (c3), (c4) we have

\[ \lim_{n \rightarrow \infty} f(y_{\sigma(n)}) = f(L) > 0 \]

This implies that

\[ f(y_{\sigma(n)}) = f(L) > 0 \quad \text{for all} \quad n \]

Then there exists an integer \( n_3 \geq n_3 \) and from equation (1), we obtain

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) - \frac{1}{2} q_{\sigma(n)} f(L) \leq 0 \quad \text{for all} \quad n \geq n_4 \]

Summing the inequality (7) from \( n_3 \) to \( n - 1 \), we have

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) - \frac{1}{2} \sum_{k=0}^{n-1} q_{\sigma(k)} f(L) \leq 0 \quad \text{for all} \quad n \geq n_4 \]

In view of (H1), (H2) from inequality (8), we find that

\[ \infty \leq 0 \quad \text{as} \quad n \rightarrow \infty \]

which is a contradiction.

Case II

\( L = \infty \)

In view of (H3), there exists an integer \( n_4 \geq n_1 \) and \( k_i > 0 \) such that \( f(y_{n_0}) \geq k_i \) for all \( n \geq n_4 \)

Therefore, from equation (1), we obtain

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) - q_k \leq 0 \quad \text{for all} \quad n \geq n_5 \]

The remaining proof is similar to that of case (I), and hence we omitted.

Thus in both cases we obtained that \( \{y_n\} \) is oscillatory.

In fact \( y_n < 0, y_{n-m} < 0 \) for all large \( n \), the proof is similar, and hence we omitted.

This completes the proof.

Corollary 1

In addition to the conditions (c1), (c2), (c3), (c4), if the conditions of theorem 1 hold. Then every bounded solution of equation (1) is oscillatory.

Proof

Proceeding as in the proof of theorem 1 with assumption that is \( \{y_n\} \) bounded non-oscillatory solution (1).

Therefore, from inequality (7) of theorem 1, we find that

\[ R_n \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) - \frac{1}{2} R_n q_{\sigma(n)} f(L) \leq 0 \quad \text{for all} \quad n \geq n_4 \]

By the definition of \( R_n \) and from the inequality (10) we find that:

\[ R_n \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) - \frac{1}{2} R_n q_{\sigma(n)} f(L) \leq 0 \quad \text{for all} \quad n \geq n_4 \]

In view of, (H1), (H2) and (c3), we have

\[ \Delta \left( \frac{1}{a_n} \Delta^2 z_n \right) \geq 0 \quad \text{for all} \quad n \]

This shows that sequence \( \{y_n\} \) is a bounded oscillatory solution of equation (1).

This completes the proof.

Theorem (A):

Let \( a_n = p_n = 1 \) and \( f \) be non-decreasing.
If \( \sum_{n=0}^{\infty} n^2 |q_n| < \infty \) then equation (1) has a nonoscillatory solution that approaches a nonzero real number as \( n \to \infty \).

**ASYMPTOTIC BEHAVIOR**

In this section, we obtain a sufficient condition for the asymptotic behavior of solutions of equation (1). We do not require \( q_n > 0 \) here. Let \( A_n, B_n \), and \( C_n \) be defined by

\[
A_n = \sum_{i=0}^{n-1} \frac{1}{d_i}, \quad B_n = \sum_{i=0}^{n-1} \frac{1}{p_i} \quad \text{and} \quad \sum_{i=0}^{n-1} A_i = B_i.
\]

**Theorem 2**

Let \( f(u) \) be non-decreasing and let \( d > 0 \) be a constant such that \( a_n \geq d \) for all \( n \geq n_a \).

Suppose that

\[
\sum_{n=n_a}^{\infty} \left| C_{n+1} + A_{n+1} B_{n+1} \right| < \infty
\]


REFERENCES


