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OSCILLATORY AND ASYMPTOTIC SOLUTIONS OF FOURTH ORDER NON-LINEAR DIFFERENCE EQUATIONS WITH DELAY

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ABSTRACT

The objective of this paper is to study the oscillatory and asymptotic solutions of fourth order nonlinear delay difference equation of the form

$$\Delta\left(\frac{1}{a_n}(\Delta^3(\mathrm{y_n} - P_n Y_{n-k})\right) - qn + 1 f(\mathrm{y}\,\sigma_{(\mathrm{n})}) = 0 \quad \text{----- (1)}$$

Example is given to illustrate the results.

INTRODUCTION

In this paper, we study the oscillatory and asymptotic behavior of solution of fourth order nonlinear delay difference equation of the form

$$\Delta\left(\frac{1}{a_n}(\Delta^3(\mathrm{y_n} - P_n Y_{n-k})\right) - q_{n+1} f(\mathrm{y}_{\sigma(\mathrm{n})}) = 0 \qquad (1)$$

Here Δ is the forward difference operator and defined by $\Delta y_n = y_{n+1} - y_n$ where k is a fixed nonnegative integer and $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are sequence of nonnegative integers with respect to the difference equation (1) throughout. A nontrivial solution $\{y_n\}$ of equation (1) is said to be oscillatory if for any $N \geq n_o$ there exists $n > N$ such that $y_{n+1} y_n \leq 0$. Otherwise, the solution is said to be non-oscillatory (Agarwal, 1992; Artzrouni, 1985; Cheng and Patula, 1993; Peterson, 1995; Philos and Purnaras, 2001) We shall assume that the following conditions hold:

(c_1) $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are real sequences and $a_n \leq 0$ for infinitely many values of n.

(c_2) f: R→R is continuous and yf(y)>0, for all y ≠ 0.

(c_3) σ (n) ≥ 0 is an integer such that $\lim_{n\to\infty} \sigma(n) = \infty$

(c_4) $\mathrm{R_n} = \Sigma_{s=n_n}^{n-1} a_s \to \infty \, as \, n \to \infty$

MAIN RESULTS

Theorem 1

In addition to the conditions

(c_1), (c_2), (c_3), (c_4), if the conditions are

$(H1)$ $\mathrm{P_n} \geq 0 \, and \, \Sigma_{s=n_o}^{\infty} p_s = \infty$

(H_3) $\lim_{(|u|\to 8)} \inf|f(u)| \geq 0$

(H_3) $\lim_{|u|\to\infty} \inf \mathrm{f}\,|f(u)| \geq 0$

Then every solution of equation (1) is oscillatory.

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Proof

Suppose that the equation (1) has non-oscillatory solution $\{y_n\}$ is eventually positive. Then there is a positive integer n_o such that $y_{\sigma(n)} \geq 0$, f or $n \geq n_o$ implies that $\{y_n\}$ is non-oscillatory. Without loss of generality we can assume that there exists an integer $n_1 \geq n_o$ such that

$y_n > 0, \Delta y_n > 0, y_{n-m} \geq 0, \Delta y_{n-m} \geq 0$ *for all* $n \geq n_1$ Set then view of (H_1),

$z_n > 0, \Delta z_n \geq 0$, for all $n = n_1$.

From equation (1) we have

$$\Delta(\frac{1}{a_n}\Delta^3 z_n) = q_{(n+1)} f(y_{(\sigma(n))}) \text{ for all } n \geq n_1 \quad (2)$$

In view of the conditions

(c_2), (c_3), (H_2) and from the equation (2), we obtain $\Delta(\frac{1}{a_n}\Delta^3 z_n)$ is eventually non-increasing.

We first show that $\Delta(\frac{1}{a_n}\Delta^3 z_n) \geq 0$ for $n \geq n_1$.

Suppose that, there exists an integer $n_2 \geq n_1$ and $k_1 > 0$ such that

$$\Delta(\frac{1}{a_n}\Delta^3 z_n) \leq -k_1 \text{ for all } n \geq n_2 \quad (3)$$

Summing the inequality (3) from n_2 to $n-1$ we have

$$\frac{1}{a_n}\Delta^3 z_n - \frac{1}{a_{n_1}} \leq -k_1, (n - n_2) \text{ for all } n \geq n_2 \quad (4)$$

Therefore $\frac{1}{a_n}\Delta^3 z_n \to -\infty \, as \, n \to \infty$

Then there exists an integer $n_2 \geq n_1$ and $k_2 > 0$ such that

$$\frac{1}{a_n}\Delta^3 z_n \to -\infty \, as \, n \to \infty \quad (5)$$

Summing the inequality (5) from n_3 to $n-1$, we have

$$\Delta^2 z_n \leq -k_2 \sum_{s=n_3}^{n-1} as, \, for \, all \, n \geq n_3 \quad (6)$$

In view of the condition (c_4), and from the inequality (5), we obtain $\Delta z_n \to -\infty \, as \, n \to \infty$ which is a contradiction to the fact that $\Delta_{zn} \geq 0$, for all large n. This shows that $\Delta(\frac{1}{a_n}\Delta^2 z_n) \geq 0$

For all large n.

Let $L = \lim_{n\to\infty} y_n$. Then L is finite or infinite.

Case 1

$L > 0$ is finite.

In view of (c_2), (c_3) we have

$\lim_{n\to\infty} f(y_{\sigma(n)}) = f(L) > 0$

This implies that

$f(y_{\sigma(n)}) = f(L) > 0$, for all n

Then there exists an integer $n_4 \geq n_3$ and from equation (1), we obtain

$$\Delta(\frac{1}{a_n}\Delta^3 z_n) - \frac{1}{2} q_n f(L) \leq 0, \text{ for all } n \geq n_4 \quad (7)$$

Summing the inequality (7) from n_4 to $n-1$, we have

$$\Delta\left(\frac{1}{a_n}\Delta^3 z_n\right) - \Delta\left(\frac{1}{a_n}\Delta^3 z_{n_3}\right) - \frac{1}{2}\sum_{s=n4}^{n-1} q_s f(L) \leq 0 \text{ for all } n \geq n_4 \quad (8)$$

In view of (H_2), (H_3) from inequality (8), we find that $\infty \leq 0$, as $n \to \infty$ which is a contradiction.

Case II

$L = \infty$

In view of (H_2), there exists an integer $n_4 \geq n_3$ and $k_3 > 0$ such that $f(y_{\sigma(n)}) > k_3$, for all $n \geq n_5$

Therefore, from equation (1), we obtain

$$\Delta\left(\frac{1}{a_n}\Delta^3 z_n\right) - q_n k_3 \leq 0, \text{ for all } n \geq n_5 \quad (9)$$

The remaining proof is similar to that of case (I), and hence we omitted.

Thus in both cases we obtained that $\{y_n\}$ is oscillatory.

In fact $y_n < 0$, $y_{n-m} < 0$ for all large n, the proof is similar, and hence we omitted.

This completes the proof.

Corollary 1

In addition to the conditions (c_1), (c_2), (c_3), (c_4), if the conditions of theorem 1 hold. Then every bounded solution of equation (1) is oscillatory.

Proof

Proceeding as in the proof of theorem 1 with assumption that is $\{y_n\}$ bounded non-oscillatory solution (1).

Therefore, from inequality (7) of theorem 1, we find that

$$R_n\Delta(\frac{1}{a_n}\Delta^3 z_n) - \frac{1}{2} R_{n+1} q_{n+1} f(L) \leq 0, \text{ for all } n \geq n_4 \quad (10)$$

By the definition of R_n and from the inequality (10) we find that:

$$R_n\Delta(\frac{1}{a_n}\Delta^3 z_n) - \frac{1}{2} R_{n+1} q_{n+1} f(L) \text{ for all } n \geq n_4 \quad (11)$$

In view of, (H_2), (H_3) and (c_4), we have

$\Delta\left(\frac{1}{a_n}\Delta^3 z_n\right) \geq 0$ for all large n.

This shows that sequence $\{y_n\}$ is a bounded oscillatory solution of equation (1).

This completes the proof.

Theorem (A):

Let $a_n = p_n \equiv 1$ and f be non-decreasing.

If $\sum_{n=n_0}^{\infty} n^2 |q_n| < \infty$ then equation (1) has a non-oscillatory solution that approaches a nonzero real number as n→∞.

ASYMPTOTIC BEHAVIOR

In this section, we obtain a sufficient condition for the asymptotic behavior of solutions of equation (1). We do not require $q_n > 0$ here. Let A_n, B_n, and C_n be defined by

$$A_n = \sum_{s=0}^{n-1} \frac{1}{a_s}, B_n = \sum_{s=0}^{n-1} \frac{1}{p_s} \; and \; \sum_{s=0}^{n-1} \frac{A_s}{B_s},$$

Theorem 2

Let f(u)be non-decreasing and let d>0 be a constant such that $a_n \geq d$ for all $n \geq n_{o.}$

Suppose that

$$\sum_{n=n_0}^{\infty} \left| C_{n+1} + A_{n+1} B_{n+1} \right| < \infty$$

Then equation (1) has a bounded non-oscillatory solution that approaches a nonzero limit (Philos, 2005; Philos, 2004; Philos, 2004; Kordonis, 2004; Philos and Purnaras, 2004).

Proof

Let c>0 and let N be so large that

$$\sum_{n=n_0}^{\infty} \left| C_{n+1} + A_{n+1} B_{n+1} \right| < \frac{c}{4f(2c)}$$

Let the Banach space β_N and the set $\mu \subseteq \beta_N$ be the same as in theorem (A) and define the operator T: $\mu \rightarrow \beta_N$ by

$$(Ty)_n = \frac{3c}{2} \sum_{s=n}^{\infty} K(s,n)(q_s f(y_{\sigma 9(n)}), n \geq N$$

Where

$K(s, n) = C_{s+1} - C_n + A_{s+1} B_n - A_{s+1} B_{s+1}$.

CONCLUSION

Similar to the proof of theorem (A), we can show that the mapping T satisfies the hypotheses of Schauder's fixed point theorem (Philos and Purnaras, 2005; Philos and Purnaras, 2004; Julio, 2005; Philos and Purnaras, 2008; Philos and Purnaras, 2010).

Hence, T has a fixed point Y∈μ, and it is clear that Y= $\{y_n\}$is a non-oscillatory solution of equation (1) for n ≥ Nand has the desired properties.

It should be pointed out that Theorem (A) is actually a special case of the above result. We conclude this paper with a simple example of Theorem (2).

Example:

$$\Delta(n\Delta^3 y_n) + (-1)^n 3^{-n} y_{n-m} = 0, \; n \geq 1 \qquad (13)$$

Where m is a positive integer. All conditions Theorem (2) are satisfied, so equation (13) has a bounded non-oscillatory solution that approaches a non-zero limit.

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